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# HOMOLOGICAL LOCALISATION OF MODEL CATEGORIES

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Abstract. One of the most useful methods for studying the stable homotopy category is localising at some spectrum $E$. For an arbitrary stable model category we introduce a candidate for the $E$–localisation of this model category. We study the properties of this new construction and relate it to some well–known categories.

## 1. Introduction

The stable homotopy category is spectacularly complicated and yet of fundamental importance to homotopy theorists. A standard and highly successful method of dealing with this complexity is to "filter out" some of this information via a Bousfield localisation. In return we obtain a more structured category with useful and interesting patterns.

More precisely, we choose some homology theory $E_*$ and replace the stable homotopy category $\mathrm{Ho}(\mathcal{S})$ with $\mathrm{Ho}(L_E\mathcal{S})$, the full subcategory of $\mathrm{Ho}(\mathcal{S})$ with objects the $E$–local spectra. This means that in the passage from $\mathrm{Ho}(\mathcal{S})$ to $\mathrm{Ho}(L_E\mathcal{S})$, the $E_*$–isomorphisms are formally inverted. Bousfield's paper [Bou79] is the original source of this idea.

There are a number of other model categories whose homotopy categories share many of the properties of $\mathrm{Ho}(\mathcal{S})$, namely stable model categories. It would be advantageous if we could generalise the notion of $E$–localisation to this class of categories. Thus we are interested in the construction of a *homological localisation* of a *stable* model category, one that is the analogue of forming $\mathrm{Ho}(L_E\mathcal{S})$ from $\mathrm{Ho}(\mathcal{S})$.

The main motivation comes again from the study of the stable homotopy category. In order to understand spectra, $\mathrm{Ho}(\mathcal{S})$ and its various $E$–localisations it is necessary to relate $\mathcal{S}$ and $L_E\mathcal{S}$ to other stable model categories $\mathcal{C}$. For example, one can study to what extent there is a stable model category $\mathcal{C}$ whose homotopy category "models" $\mathrm{Ho}(L_E\mathcal{S})$ and how similar $\mathcal{C}$ is to $L_E\mathcal{S}$ in terms of higher homotopy behaviour. To make those links it would be a desirable tool to have the corresponding $E$–localisations of $\mathcal{C}$ in order to compare $E$–local spectra to other counterparts related to $\mathcal{C}$.

A stable model category $\mathcal{C}$ is a model category whose associated homotopy category $\mathrm{Ho}(\mathcal{C})$ is triangulated via the construction of [Hov99, Section 7]. Lenhardt proved in [Len12] that $\mathrm{Ho}(\mathcal{C})$ is a module over $\mathrm{Ho}(\mathcal{S})$ whenever $\mathcal{C}$ is a stable model category. Hence we have a tensor product

$$- \wedge^L - \colon \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

and an enrichment of $\mathrm{Ho}(\mathcal{C})$ in $\mathrm{Ho}(\mathcal{S})$. This technique is called *stable frames.*

Using this action on the homotopy category of a stable model category one could try to make a new model structure on $\mathcal{C}$ such that the weak equivalences are the "$E_*$–isomorphisms": those maps $f\colon X \to Y$ in $\mathcal{C}$ such that

$$f \wedge^L E \colon X \wedge^L E \to Y \wedge^L E$$

is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. Such a model structure would deserve the name $L_E\mathcal{C}$. The machinery that allows one to create new model structures with larger collections of weak

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equivalences is Bousfield localisation, see [Hir03, Part I]. But it seems particularly difficult to check that $L_E\mathcal{C}$ exists for general $\mathcal{C}$. For spectra, the argument appears in [EKMM97, Section VIII.1] and requires numerous unpleasant cardinality arguments.

For well–behaved stable model categories $\mathcal{C}$ we are going to produce a new model structure $\mathcal{C}_E$ that avoids such set–theoretic awkwardness. This $\mathcal{C}_E$ is a good candidate for the $E$–localisation of $\mathcal{C}$ because of the following universal property: $\mathcal{C}_E$ is the "closest" model category to $\mathcal{C}$ such that any Quillen adjunction from spectra to $\mathcal{C}$

$$\mathcal{S} \rightleftarrows \mathcal{C}$$

gives rise to a Quillen adjunction

$$L_E\mathcal{S} \rightleftarrows \mathcal{C}_E$$

from $E$–local spectra to $\mathcal{C}_E$. We are also able to give another description of $\mathcal{C}_E$ in terms of pushouts of model categories, which shows how strong the universal property of this new model structure is.

We are also able to give an improvement of [BR11, Theorem 9.5]: we can show that for all $E$, the homotopy information of $E$–local spectra is entirely encoded in the $\mathrm{Ho}(\mathcal{S})$–module structure on the $E$–local stable homotopy category. This was previously only possible with the strong restriction that $E$ is smashing. Hence we have the following, which appears as Theorem 9.1.

**Theorem 1.** *Let $\mathcal{C}$ be a stable model category. Assume we have an equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

*then $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent if and only if $\Phi$ is an equivalence of $\mathrm{Ho}(\mathcal{S})$–module categories.*

## 2. Organisation

Firstly, we recall some definitions and conventions regarding Bousfield localisation and stable frames. We also re–introduce the concept of *stably $E$–familiar model categories*: in [BR11] we studied those $\mathcal{C}$ such that the action of $\mathrm{Ho}(\mathcal{S})$ factors over the functor $\mathrm{Ho}(\mathcal{S}) \to \mathrm{Ho}(L_E\mathcal{S})$. In particular the homotopy category of such a model category has an enrichment in the more structured category $\mathrm{Ho}(L_E\mathcal{S})$. We called such categories stably $E$–familiar.

We then turn to the question of altering a model structure on a given category so as to obtain a stably $E$–familiar model category. In Section 5 we consider the simpler case of spectral model categories: such a model category is defined in a similar way to a simplicial model category, but with simplicial sets replaced by the model category of symmetric spectra. We construct the *stable $E$–familiarisation* of a spectral model category in this section.

In Section 6 we extend our results to more general stable model categories. We prove that the *stable $E$–familiarisation* of a model category $\mathcal{C}$ is the closest stably $E$–familiar model category to $\mathcal{C}$ in the following sense. The result below also implies that our construction has the universal property we described earlier.

**Theorem 2.** *Let $\mathcal{C}$ be a stable, proper and cellular model category such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. Then there is a model structure $\mathcal{C}_E$ on $\mathcal{C}$ such that*

(1) *$\mathcal{C}_E$ is stably $E$–familiar,*
(2) *if $F : \mathcal{C} \to \mathcal{E}$ is a left Quillen functor and $\mathcal{E}$ is stably $E$–familiar, then $F$ factors over $\mathcal{C} \to \mathcal{C}_E$.*

Section 7 consists of several *examples* of $\mathcal{C}_E$ for some $E$ and $\mathcal{C}$ involving algebraic model categories, chromatic localisations and module categories over a ringoid spectrum.

In Section 8 we rephrase the universal property of $\mathcal{C}_E$ in terms of *homotopy pushouts* of model categories.

Finally, we prove a full version of the *modular rigidity* theorem that all homotopy information of $E$–local spectra is governed by the $\mathrm{Ho}(\mathcal{S})$–action on $\mathrm{Ho}(L_E\mathcal{S})$ given by framings.

## 3. Bousfield localisation

We begin with an introduction to Bousfield localisation at a homology theory $E$. Throughout the paper when we refer to spectra, we mean symmetric spectra equipped with the stable model structure [HSS00] unless stated otherwise.

Let $E$ be a spectrum and let $[-,-]_*$ denote maps in the stable homotopy category. Then $E$ corepresents a homology functor $E_*$ on the category of spectra via

$$E_*(X) = [S^0, E \wedge X]_*$$

where $S^0$ denotes the sphere spectrum. Bousfield used this to construct a homotopy category of spectra where maps which induce isomorphisms on $E_*$–homology are isomorphisms [Bou79]. We recap some of the definitions from this work.

**Definition 3.1.** *A map $f\colon X \to Y$ of spectra is an $E$–*equivalence *if $E_*(f)$ is an isomorphism. A spectrum $Z$ is $E$–*local *if $f^*\colon [Y,Z] \to [X,Z]$ is an isomorphism for all $E$–equivalences $f\colon X \to Y$. A spectrum $A$ is $E$–*acyclic *if $[A,Z] = 0$ for all $E$–acyclic $Z$. An $E$–equivalence from $X$ to an $E$–local object $Z$ is called an $E$–*localisation*.*

Bousfield localisation of spectra gives a homotopy theory that is particularly sensitive towards $E_*$ and $E$–local phenomena. The $E$–local homotopy theory is obtained from the category of spectra by formally inverting the $E$–equivalences.

This can be seen as a special case of a more general result by Hirschhorn. Let $\mathcal{C}$ be a model category. For $X, Y \in \mathcal{C}$, we let $\mathrm{map}_{\mathcal{C}}(X,Y) \in \mathrm{sSet}$ denote the homotopy function object, see [Hir03, Chapter 17] and Section 4.

**Definition 3.2.** *Let $S$ be a class of maps in $\mathcal{C}$. Then an object $Z \in \mathcal{C}$ is $S$–*local *if*

$$\mathrm{map}_{\mathcal{C}}(s,Z) : \mathrm{map}_{\mathcal{C}}(B,Z) \longrightarrow \mathrm{map}_{\mathcal{C}}(A,Z)$$

*is a weak equivalence in simplicial sets for any $s : A \longrightarrow B$ in $S$. A map $f : X \longrightarrow Y \in \mathcal{C}$ is an $S$–*equivalence *if*

$$\mathrm{map}_{\mathcal{C}}(f,Z) : \mathrm{map}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{map}_{\mathcal{C}}(X,Z)$$

*is a weak equivalence for any $S$–local $Z \in \mathcal{C}$. An object $W \in \mathcal{C}$ is $S$–*acyclic *if*

$$\mathrm{map}_{\mathcal{C}}(W,Z) \simeq *$$

*for all $S$–local $Z \in \mathcal{C}$.*

A *left Bousfield localisation* of a model category $\mathcal{C}$ with respect to a class of maps $S$ is a new model structure $L_S\mathcal{C}$ on $\mathcal{C}$ such that

- the weak equivalences of $L_S\mathcal{C}$ are the $S$–equivalences,
- the cofibrations of $L_S\mathcal{C}$ are the cofibrations of $\mathcal{C}$,
- the fibrations of $L_S\mathcal{C}$ are those maps that have the right lifting property with respect to cofibrations that are also $S$–equivalences.

Hirschhorn proves that if $S$ is a set and $\mathcal{C}$ is left proper and cellular then $L_S\mathcal{C}$ exists. (We will give rough definitions of these two terms below.) Note that an object is fibrant in $L_S\mathcal{C}$ if and only it is fibrant in $\mathcal{C}$ and $S$–local.

In the case of localising spectra at a homology theory one wants to invert the class of $E_*$–isomorphisms, i.e. those maps of spectra that induce isomorphisms in the homology theory $E_*$. Since this is not a set, one cannot use Hirschhorn's result directly. In [EKMM97, Section VIII.1] it is shown that there is a set $S$ whose $S$–equivalences are exactly the $E_*$–isomorphisms. Hence, the key to proving the existence of homological localisations is to find a set giving the correct notion of equivalence. We shall encounter this idea again when constructing $\mathcal{C}_E$.

A model category is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. A model category is *right proper* if the pullback of a weak equivalence along a fibration is a weak equivalence. If a model category is both left and right proper, we say that it is *proper*.

We also a need a stronger version of "cofibrantly generation", one which forces cell complexes to be better behaved. The actual definition is technical and not particularly illuminating, so we shall simply say that a model category is *cellular* if it is cofibrantly generated by sets $I$ and $J$, and the domains and codomains of $I$ and $J$ satisfy some nice cardinality conditions. We leave the details to [Hir03, Definition 12.1.1].

## 4. Stable framings

Framings are a powerful tool that describe and classify Quillen functors from simplicial sets or spectra to arbitrary model categories. They were first developed by Hovey in [Hov99, Section 5.2]. For a model category $\mathcal{C}$, he investigates cosimplicial and simplicial resolutions of objects in $\mathcal{C}$. These are called "frames". In more detail, a frame of an object $A \in \mathcal{C}$ is a cofibrant replacement of the constant cosimplicial object $A \in \mathcal{C}^\Delta$ in the Reedy model category of cosimplicial objects in $\mathcal{C}$. From these notions one obtains bifunctors

$$\begin{aligned} -\otimes - &: \mathcal{C} \times \mathrm{sSet} \longrightarrow \mathcal{C}, \\ \mathrm{map}_l(-,-) &: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{sSet}, \\ (-)^{(-)} &: \mathcal{C} \times \mathrm{sSet}^{op} \longrightarrow \mathcal{C}, \\ \mathrm{map}_r(-,-) &: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{sSet} \end{aligned}$$

satisfying certain adjunction properties. The notation $\otimes$ stems from the fact that

$$A \otimes \Delta[0] \simeq A.$$

However, this set-up does not equip $\mathcal{C}$ with the structure of a simplicial model category because the "mapping spaces" $\mathrm{map}_l(X,Y)$ and $\mathrm{map}_r(X,Y)$ only agree up to a zig-zag of weak equivalences for cofibrant $X \in \mathcal{C}$ and fibrant $Y \in \mathcal{C}$ [Hov99, Proposition 5.4.7]. But their derived functors agree, leaving us with the following [Hov99, Theorem 5.5.3].

**Theorem 4.1** (Hovey)**.** *Let $\mathcal{C}$ be any model category. Then its homotopy category* $\mathrm{Ho}(\mathcal{C})$ *is a closed* $\mathrm{Ho}(\mathrm{sSet})$*–module category.*

In particular, this equips any model category with the notion of a homotopy mapping space. Moreover, framings satisfy the following important properties.

- If $\mathcal{C}$ carries the structure of a simplicial model category [Hov99, Definition 4.2.18], then the two Ho(sSet)–module structures coming from either framings or the simplicial structure agree [Hov99, Theorem 5.6.2].
- If $F : \mathrm{sSet} \longrightarrow \mathcal{C}$ is a left Quillen functor with $F(\Delta[0]) = A$, then the left derived functors of $F$ and of the framing functor $A \otimes - : \mathrm{sSet} \longrightarrow \mathcal{C}$ agree. Thus, every left Quillen functor from simplicial sets to any model category can be described, up to homotopy, by a frame.

The second property follows from the fact that the category of cosimplicial objects $\mathcal{C}^\Delta$ is equivalent to the category of adjunctions sSet $\rightleftarrows \mathcal{C}$ [Hov99, Proposition 3.1.5]. A cosimplicial object $A^\bullet$ corresponds to a Quillen adjunction under this equivalence if and only if it is a frame, that is $A^\bullet$ is cofibrant and homotopically constant, [BR11, Proposition 3.2].

In [Len12] Fabian Lenhardt described an analogous set-up for spectra and stable model categories. Now let $\mathcal{C}$ be a stable model category. First, Lenhardt shows that the category of adjunctions between spectra and a stable model category $\mathcal{C}$ is equivalent to the category of "$\Sigma$–cospectra" $\mathcal{C}^\Delta(\Sigma)$. An object in $\mathcal{C}^\Delta(\Sigma)$ consists of a sequence of cosimplicial objects $X_n \in \mathcal{C}^\Delta$ together with structure maps

$$\Sigma X_{n+1} \longrightarrow X_n.$$

The suspension of cosimplicial objects is described in [Len12, Section 3.3]. He then characterises those $\Sigma$–cospectra that give rise to Quillen adjunctions under this equivalence, calling them stable frames. These give rise to bifunctors $- \wedge -$ and $\mathrm{Map}(-,-)$ satisfying the expected adjunction properties.

As in the unstable case, this is not rigid enough to equip any stable model category $\mathcal{C}$ with the structure of a spectral model category. However, the above bifunctors give rise to the following [Len12, Theorem 6.3].

**Theorem 4.2** (Lenhardt)**.** *Let $\mathcal{C}$ be a stable model category. Then $\mathrm{Ho}(\mathcal{C})$ is a closed $\mathrm{Ho}(\mathcal{S})$–module category.*

As expected, this satisfies the following key properties.

- If $\mathcal{C}$ is already a spectral model category, then the $\mathrm{Ho}(\mathcal{S})$–module structure derived from the spectral structure agrees with the $\mathrm{Ho}(\mathcal{S})$–module structure coming from stable frames [BR11, Example 6.7].
- By construction, every left Quillen functor $F : \mathcal{S} \longrightarrow \mathcal{C}$ is, up to homotopy, of the form $X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$ for some fibrant–cofibrant $X \in \mathcal{C}$.
- In particular, for any fibrant–cofibrant $X \in \mathcal{C}$ there is a left Quillen functor $\mathcal{S} \longrightarrow \mathcal{C}$ that sends the sphere spectrum to $X$. We denote this functor by $X \wedge -$ and its right adjoint by $\mathrm{Map}_\mathcal{C}(X,-)$.
- Any stable frame and thus any Quillen functor $\mathcal{S} \longrightarrow \mathcal{C}$ is, up to homotopy, entirely determined by its image on the sphere.

As we have already mentioned, the homotopy theory of $L_E\mathcal{S}$ is often much better understood than $\mathcal{S}$. So it is worth asking if some stable model categories have more in common with $L_E\mathcal{S}$ than $\mathcal{S}$. We answer this question and obtain several useful results using this idea in [BR11]. We give the fundamental definitions below.

**Definition 4.3.** *We say that a stable frame $X \in \mathcal{C}^\Delta(\Sigma)$ is an $E$–*local frame *if it gives rise to a Quillen functor pair*

$$X \wedge - : L_E\mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Map}_\mathcal{C}(X,-).$$

*A stable model category $\mathcal{C}$ is* stably $E$–familiar *if every stable frame is an $E$–local frame.*

This is [BR11, Definition 7.1]. This generalises the notion of an $L_E\mathcal{S}$–model category in the following sense: if $\mathcal{C}$ is already an $L_E\mathcal{S}$–model category, then the $\mathrm{Ho}(L_E\mathcal{S})$–module structure on $\mathrm{Ho}(\mathcal{C})$ agrees with the $\mathrm{Ho}(L_E\mathcal{S})$–module structure given by $E$–local frames [BR11, Proposition 7.6]. We can further characterise stably $E$–familiar model categories as follows [BR11, Theorem 7.8].

**Theorem 4.4.** *Let $\mathcal{C}$ be a stable model category. Then $\mathcal{C}$ is stably $E$–familiar if and only if the homotopy mapping spectrum $\mathbb{R}\,\mathrm{Map}_\mathcal{C}(X,Y)$ is an $E$–local spectrum for all $X, Y \in \mathcal{C}$.*

We can use the theory of $E$–local framings to study *algebraic model categories*. An algebraic model category is a $\mathrm{Ch}(\mathbb{Z})$–model category in the sense of [Hov99, Definition 4.2.18]. Thus a $\mathrm{Ch}(\mathbb{Z})$–model category is enriched, tensored and cotensored over chain complexes and satisfies the $\mathrm{Ch}(\mathbb{Z})$–analogue of the compatibility axiom (SM7). This implies that the homomorphism spectra obtained via framings are products of Eilenberg–Mac Lane spectra [GJ99, Proposition III.2.20], [DS07, Proposition 1.6]. Using the computations of Gutiérrez in [Gut10] one can draw the following conclusions [BR11, Section 9].

- For $n \geq 1$ there are no algebraic stably $K(n)$–familiar model categories, where $n$ denotes the $n^{th}$ Morava–$K$–theory.
- Let $E(n)$ denote the $n^{th}$ chromatic Johnson–Wilson spectrum. An algebraic model category is stably $E(n)$–familiar if and only if it is rational.

Now we turn to the question of whether any model category can be made stably $E$–familiar in some natural way.

## 5. $E$–Familiarisation of spectral model categories

For any homology theory $E$ we can consider the category of spectra with the $E$–local model structure, $L_E\mathcal{S}$. Hence we would like to know if a reasonable notion of $E$–localisation exists for an arbitrary stable model category $\mathcal{C}$.

Intuitively, a promising definition would be a Bousfield localisation $L_E\mathcal{C}$ of $\mathcal{C}$ where one localises at the class of "$E$–equivalences" given by

$$\{f : X \longrightarrow Y \in \mathcal{C} \mid f \wedge^L E : X \wedge^L E \longrightarrow Y \wedge^L E \text{ is an isomorphism in } \mathrm{Ho}(\mathcal{C})\},$$

where the action $\wedge$ of a spectrum on an element of $\mathcal{C}$ is defined via stable frames. However, showing the existence of Bousfield localisations at a class of maps is set–theoretically awkward. The standard method to circumvent this difficulty is to find a set of maps $S$ such that the $S$–equivalences are precisely the $E$–equivalences. This is an extremely difficult task, see [EKMM97, Section VIII.1], so it is not clear if a good notion of $E$–localisation exists for general model categories.

Instead, we will construct the *stable $E$–familiarisation* $\mathcal{C}_E$ of $\mathcal{C}$ which is the "closest" stably $E$–familiar model category to $\mathcal{C}$. We will then draw some conclusions about its properties which will show that this construction is the right choice for an analogue of $E$–localisation for general stable $\mathcal{C}$. For example, the first theorem will show that every Quillen adjunction

$$\mathcal{S} \rightleftarrows \mathcal{C}$$

will give rise to a Quillen adjunction

$$L_E\mathcal{S} \rightleftarrows \mathcal{C}_E.$$

The first question to answer is: what kind of maps do we want to invert in order to construct $\mathcal{C}_E$? In a stably $E$–familiar model category $\mathcal{D}$ any map of the form

$$X \wedge^L j \colon X \wedge^L A \to X \wedge^L B$$

for $j \colon A \to B$ an $E$–equivalence of spectra and $X \in \mathcal{D}$ is a weak equivalence. Hence we could try to localise $\mathcal{C}$ at this class of maps. So we must find some set of maps $S$ such that the $S$–equivalences equals this class.

We need a couple of technical results first. For this section we shall work with $\mathcal{S}$–model categories in the sense of [Hov99, Definition 4.2.18], where $\mathcal{S}$ again denotes the model category of symmetric spectra. Such a model category $\mathcal{D}$ is enriched, tensored and cotensored over symmetric spectra in simplicial sets and satisfies the appropriate analogue of Quillen's (SM7) axiom for simplicial model categories. We shall refer to $\mathcal{D}$ as being a *spectral model category*. We may also talk about $L_E\mathcal{S}$–model categories, where we use the $E$–local model structure on $\mathcal{S}$. A spectral model category is in particular stable and simplicial, see [SS03, Lemma 3.5.2]. We will see later that the restriction to spectral model categories is not as big a restriction as it might seem at first.

We denote the pushout–product of two maps by $\square$, so for $f : X \longrightarrow Y$ and $g : A \longrightarrow B$ the pushout–product of $f$ and $g$ is

$$f \square g : X \wedge B \coprod_{X \wedge A} Y \wedge A \longrightarrow Y \wedge B.$$

Recall that a set of maps $S$ in a stable model category $\mathcal{D}$ is said to be *stable* if the class of $S$–local objects is closed under suspension. By [BR14, Proposition 4.6] if $\mathcal{D}$ and $S$ are stable then so is $L_S\mathcal{D}$.

**Proposition 5.1.** *Let $\mathcal{D}$ be a left proper, cellular and spectral model category. Let $S$ be a stable set of maps in $\mathcal{D}$. Then $L_S\mathcal{D}$ is also a spectral model category.*

*Proof.* Since $\mathcal{D}$ is left proper and cellular, $L_S\mathcal{D}$ exists by [Hir03, Theorem 4.1.1]. We must prove that if $i$ is a cofibration of $L_S\mathcal{D}$ and $j$ is a cofibration of $\mathcal{S}$ then $i \square j$ is a cofibration of $L_S\mathcal{D}$ that is a weak equivalence (in $L_S\mathcal{D}$) if either of $i$ or $j$ is. Since $\mathcal{D}$ is spectral and the cofibrations are unchanged by left Bousfield localisation, we know that $i \square j$ is a cofibration whenever $i$ and $j$ are. Furthermore if $j$ is an acyclic cofibration of symmetric spectra, then $i \square j$ is a weak equivalence in $\mathcal{D}$ and hence it is also an $S$–equivalence.

The third case is where $i$ is an acyclic cofibration of $L_S\mathcal{D}$ and $j$ is a cofibration of symmetric spectra. We must show that $i \square j$ is an $S$–equivalence. By [Hov99, Lemma 4.2.4] it suffices to prove this for $j$ a generating cofibration of symmetric spectra and $i$ a generating acyclic cofibration of $L_S\mathcal{D}$. By [HSS00, Proposition 3.4.2] we may assume that $j$ is of the form

$$F_n K \to F_n L$$

where $F_n$ is the left adjoint to evaluation at level $n$, and $K$ and $L$ are simplicial sets. By [Hir03, Proposition 4.5.1] the domain of $i$ is cofibrant, so it follows that both the domain and codomain of $i \square j$ are cofibrant. The set $S$ is stable, so the class of $S$–equivalences in $\mathrm{Ho}(\mathcal{D})$ is closed under suspension and desuspension. Thus $i \square j$ is an $S$–equivalence if and only if

$$\Sigma^n(i \square j) \cong i \square \Sigma^n j$$

is an $S$–equivalence for all $n$.

We know that $\Sigma^n F_n K$ is weakly equivalent to $F_0 K$ in $\mathcal{S}$. Hence for any cofibrant $X \in \mathcal{D}$,

$$X \wedge \Sigma^n F_n K \to X \wedge F_0 K$$

is a weak equivalence of $\mathcal{D}$. We also know that the domains of the maps $i \square \Sigma^n j$ and $i \square (F_0 K \to F_0 L)$ are pushouts of cofibrations between cofibrant objects. It follows that $i \square \Sigma^n j$ is weakly equivalent to the map $i \square (F_0 K \to F_0 L)$. The bifunctor

$$- \wedge F_0 - : \mathcal{D} \times \mathrm{sSet} \to \mathcal{D}$$

gives $\mathcal{D}$ the structure of a simplicial model category. We may now use [Hir03, Theorem 4.1.1], which states that since $\mathcal{D}$ is simplicial, so is $L_S\mathcal{D}$. Consequently we see that $i \square (F_0 K \to F_0 L)$ is an $S$–equivalence. Hence $i \square j$ is also an $S$–equivalence and $L_S\mathcal{D}$ is a spectral model category. □

**Proposition 5.2.** *Let $\mathcal{D}$ be a left proper, cellular and spectral model category with generating cofibrations $I_{\mathcal{D}}$ and generating acyclic cofibrations $J_{\mathcal{D}}$. Let $J_E$ be the set of generating acyclic cofibrations for $L_E\mathcal{S}$. Define*

$$S = I_{\mathcal{D}} \square J_E = \{ i \square j \mid i \in I_{\mathcal{D}}, \quad j \in J_E \}.$$

*Then $L_S\mathcal{D}$ is an $L_E\mathcal{S}$–model category and hence is stably $E$–familiar.*

*Proof.* The set $J_E$ is closed under desuspension in the sense that for any element $j \in J_E$ there is an element $j'$ with $\Sigma j' \simeq j$. It follows that the same holds for $S$, so it is stable in the sense of [BR14, Definition 3.2]. Thus $L_S\mathcal{D}$ is also a stable model category. By Lemma 5.1 it is also an $\mathcal{S}$–model category.

To see that it is an $L_E\mathcal{S}$–model category we only need to check that if $i$ is a cofibration of $L_S\mathcal{D}$ and $j$ is an acyclic cofibration of $L_E\mathcal{S}$ then $i\Box j$ is an $S$–equivalence. By [Hov99, Lemma 4.2.4] it suffices to prove this for $i \in I_{\mathcal{D}}$ and $j \in J_E$. But then $i\Box j$ is an element of $S$ and hence is an $S$–equivalence. □

**Proposition 5.3.** *Let $\mathcal{D}$ be a left proper, cellular and spectral model category and $S$ as in Proposition 5.2. Assume that the domains of the generating cofibrations of $\mathcal{D}$ are cofibrant. Then if $\mathcal{D}$ is a monoidal model category so is $L_S\mathcal{D}$.*

*Proof.* Since $\mathcal{D}$ is spectral, the maps in $S$ are cofibrations between cofibrant objects. Thus by [BR14, Lemma 6.1] $L_S\mathcal{D}$ is monoidal if and only if

$$I_{\mathcal{D}}\Box S = I_{\mathcal{D}}\Box(I_{\mathcal{D}}\Box J_E) \cong (I_{\mathcal{D}}\Box I_{\mathcal{D}})\Box J_E$$

lies in the class of $S$–equivalences. As $\mathcal{D}$ is monoidal, $I_{\mathcal{D}}\Box I_{\mathcal{D}}$ consists of cofibrations. By Proposition 5.2, $L_S\mathcal{D}$ is an $L_E\mathcal{S}$–model category. Hence the pushout product of a cofibration of $\mathcal{D}$ and an acyclic cofibration of $L_E\mathcal{S}$ is an $S$–equivalence as required. □

We now show that this set $S$ has the correct homotopical behaviour in terms of $E$–familiarity by giving another description of the weak equivalences of $L_S\mathcal{D}$.

**Proposition 5.4.** *Let $\mathcal{D}$ be a left proper, cellular spectral model category, such that the domains of the generating cofibrations of $\mathcal{D}$ are cofibrant. Let $T$ be the class of maps*

$$T = \{X \wedge^L f \mid X \in \mathcal{D},\ f \text{ is an } E\text{–equivalence of spectra}\}.$$

*Then the class of $T$–equivalences is equal to the class of $S$–equivalences.*

*Proof.* Take some cofibrant $X \in \mathcal{D}$. Then the functor

$$X \wedge - \colon L_E\mathcal{S} \to L_S\mathcal{D}$$

is a left Quillen functor by Proposition 5.2. Hence $X \wedge -$ takes $E$–equivalences between cofibrant spectra to $S$–equivalences. Thus every element of $T$ is a weak equivalence in $L_S\mathcal{D}$.

Now we will show that every element of $S$ is also a $T$–equivalence. Consider $i\Box j \in S$ for $i\colon X \to Y$ a generating cofibration of $\mathcal{D}$ and $j\colon A \to B$ a generating acyclic cofibration for $L_E\mathcal{S}$. Since $X$, $Y$, $A$ and $B$ are all cofibrant, the maps $X \wedge j$ and $Y \wedge j$ are in the class $T$. Let $P$ be the domain of $i\Box j$, then by [Hir03, Lemma 3.4.2], the map from $Y \wedge A \to P$ is also a $T$–equivalence. It follows by the two–out–of–three property that $i\Box j$ is a $T$–equivalence. □

If the category $\mathcal{D}$ is already stably $E$–familiar then the class $T$ is already contained in the category of weak equivalences. Hence so is the set $S$, and $\mathcal{D}$ is in fact an $L_E\mathcal{S}$–model category.

**Corollary 5.5.** *Let $\mathcal{D}$ be a left proper cellular spectral model category that is stably $E$–familiar. Assume that the domains of the generating cofibrations of $\mathcal{D}$ are cofibrant. Then $\mathcal{D}$ is an $L_E\mathcal{S}$–model category.* □

## 6. $E$–Familiarisation of stable model categories

We now want to consider model categories that are not necessarily spectral. Consider a proper and cellular stable model category $\mathcal{C}$. By [BR14, Theorem 8.2] $\mathcal{C}$ is Quillen equivalent to a spectral model category, namely the category $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$ of symmetric spectra in simplicial objects in $\mathcal{C}$ equipped with a non–standard model structure. Hence there is a Quillen equivalence which by abuse of notation we call

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C}) : \Omega^\infty$$

This model category $\mathcal{D}$ is also proper and cellular. Furthermore, if the generating cofibrations for $\mathcal{C}$ have cofibrant domains, then so do the generating cofibrations for $\mathcal{D}$.

**Theorem 6.1.** *Let $\mathcal{C}$ be a stable, proper and cellular model category, such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. Let $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$, with generating cofibrations $I_\mathcal{D}$ and fibrant replacement $\hat{f}$. We set $S = I_\mathcal{D} \Box J_E$ as in Proposition* 5.2.

Define $\mathcal{C}_E$ to be the left Bousfield localisation of $\mathcal{C}$ at the set of maps $\Omega^\infty \hat{f} S$. Then

(1) $\mathcal{C}_E$ is stably $E$–familiar,
(2) the weak equivalences of $\mathcal{C}_E$ are the $T'$–equivalences, for $T'$ the class below
$$T' = \{X \wedge^L f \mid X \in \mathcal{C},\ f \text{ is an } E\text{–equivalence of spectra}\}$$
(3) if $F : \mathcal{C} \to \mathcal{E}$ is a left Quillen functor and $\mathcal{E}$ is stably $E$–familiar, then $F$ factors over $\mathcal{C} \to \mathcal{C}_E$, i.e. $F : \mathcal{C}_E \longrightarrow \mathcal{E}$ is also a left Quillen functor.

*Proof.* The model categories $\mathcal{C}$ and $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$ are Quillen equivalent. Hence [Hir03, Theorem 3.3.20] tell us that the adjunction

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathcal{D} : \Omega^\infty.$$

induces a Quillen equivalence between $L_{\Omega^\infty \hat{f} S}\mathcal{C}$ and $L_{\Sigma^\infty \hat{c} \Omega^\infty \hat{f} S}\mathcal{D}$. (Here, $\hat{c}$ denotes the cofibrant replacement in $\mathcal{C}$.) The model category $L_{\Sigma^\infty \hat{c} \Omega^\infty \hat{f} S}\mathcal{D}$ is equal to $L_S\mathcal{D}$ since $(\Sigma^\infty, \Omega^\infty)$ is a Quillen equivalence. Thus we have a Quillen equivalence between $\mathcal{C}_E = L_{\Omega^\infty \hat{f} S}\mathcal{C}$ and $L_S\mathcal{D}$. The second category is stably $E$–familiar by Proposition 5.2. Hence so is $\mathcal{C}_E$ by [BR11, Lemma 7.10].

We may also conclude that the left derived functor of $\Sigma^\infty$ induces an bijection between the weak equivalences of $\mathcal{C}_E$ (considered as a class in $\operatorname{Ho}\mathcal{C}$) and the $S$–equivalences of $\operatorname{Ho}\mathcal{D}$. Proposition 5.4 tells us that the class of $S$–equivalences in $\mathcal{D}$ is equal to the class of $T$–equivalences where

$$T = \{X \wedge^L f \mid X \in \mathcal{D},\ f \text{ is an } E\text{–equivalence of spectra}\}.$$

Consider the class of maps

$$T' = \{X \wedge^L f \mid X \in \mathcal{C},\ f \text{ is an } E\text{–equivalence of spectra}\}.$$

Let $\mathbb{L}\Sigma^\infty$ and $\mathbb{R}\Omega^\infty$ denote the left and right derived functors of $\Sigma^\infty$ and $\Omega^\infty$ respectively. By [Len12, Theorem 6.3]

$$\mathbb{L}\Sigma^\infty(X \wedge^L f) = (\mathbb{L}\Sigma^\infty X) \wedge^L f.$$

Hence $\mathbb{L}\Sigma^\infty$ takes elements of $T'$ to elements of $T$. Consider some element $Y \wedge^L f$ of $T$. This is weakly equivalent to

$$(\mathbb{L}\Sigma^\infty \mathbb{R}\Omega^\infty Y) \wedge^L f$$

and hence is in $\mathbb{L}\Sigma^\infty T'$. Thus the derived functor of $\Sigma^\infty$ induces a bijection between the class $T'$ and the class $T$ up to weak equivalence. As a consequence the derived functor of $\Sigma^\infty$ induces a bijection between the class of $T'$–equivalences and the class of $T$–equivalences. It follows that the $T'$–equivalences must be the class of weak equivalences of $\mathcal{C}_E$.

For the final point, let $F\colon \mathcal{C} \to \mathcal{E}$ be a left Quillen functor. If $\mathcal{E}$ is stably $E$–familiar, then the left derived functor of $F$ takes the $T'$–equivalences to weak equivalences of $\mathcal{E}$. Hence $F\colon \mathcal{C}_E \to \mathcal{E}$ is also a left Quillen functor. □

**Remark 6.2.** *Let $\mathcal{C}$ be a stable, proper and cellular model category, such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. Then the above result says that $\mathcal{C}_E$ is the "closest" stably $E$–familiar model category to $\mathcal{C}$.*

In particular a model category $\mathcal{C}$ is stably $E$–familiar if and only if $\mathcal{C}_E = \mathcal{C}$.

**Remark 6.3.** *The assumptions on $\mathcal{C}$ are more reasonable than they might seem in practice. Since we want to perform a left Bousfield localisation, we will have to assume that $\mathcal{C}$ is left proper and cellular. To assume that $\mathcal{C}$ is also right proper is not too much of a restriction.*

*We also need another assumption: that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. This is a subtle assumption that occurs elsewhere in the literature, for example in* [Hov01]. *We note that this assumption holds for almost all of the cofibrantly generated model categories that arise naturally.*

It is easy to check that the homotopy mapping spectra for $\mathcal{C}_E$ are given by the formula below, where $Y_E$ is the fibrant replacement of $Y$ in $\mathcal{C}_E$.

$$\mathbb{R}\operatorname{Map}_{\mathcal{C}_E}(X,Y) = \mathbb{R}\operatorname{Map}_{\mathcal{C}}(X,Y_E)$$

In particular, this mapping spectrum is $E$–local. We can use this to draw some immediate consequences of $E$–familiarisation.

For example, the chromatic Johnson–Wilson theories $E(n)$ satisfy

$$L_{E(n-1)}L_{E(n)} = L_{E(n-1)}$$

[Rav92, 7.5.3]. Thus,

**Corollary 6.4.** *For a proper and cellular stable model category $\mathcal{C}$ we have*

$$(\mathcal{C}_{E(n)})_{E(n-1)} = \mathcal{C}_{E(n-1)}.$$

□

We can further use our knowledge of stably $E$–familiar algebraic model categories described at the end of Section 4 to read off the following corollaries.

**Corollary 6.5.** *Let $\mathcal{C}$ be an algebraic model category and $K(n)$ the $n^{th}$ Morava–$K$–theory for $n \geq 1$. Then $\mathcal{C}_{K(n)}$ is trivial.* □

**Corollary 6.6.** *Let $\mathcal{C}$ be an algebraic model category and let $E(n)$ denote the $n^{th}$ chromatic Johnson–Wilson spectrum. Then $\mathcal{C}_{E(n)} = \mathcal{C}_{H\mathbb{Q}}$.* □

If we assume that localisation at $E$ is smashing, we can obtain a nicer description of the weak equivalences of $\mathcal{C}_E$: in the smashing case $\mathcal{C}_E$ is precisely the "naive" localisation of $\mathcal{C}$ at $L_ES^0$ as described in the introduction of Section 5. That is, the left Bousfield localisation of $\mathcal{C}$ at the class of $L_ES^0$–equivalences (which we denote as $L_{L_ES^0}\mathcal{C}$) exists and is equal to $\mathcal{C}_E$. With this extra assumption we also see that

$$\mathcal{C}_E = \mathcal{C}_{L_ES^0}.$$

However, for a general model category $\mathcal{C}$ and smashing $E$ it is unclear whether the model category $L_E\mathcal{C}$ exists and if it would be Quillen equivalent to $L_{L_ES^0}\mathcal{C}$.

**Lemma 6.7.** *In addition to the assumptions of Theorem 6.1, assume that localisation at $E$ is smashing. Then a map $f$ in $\mathcal{C}_E$ is a weak equivalence if and only if $f \wedge^L L_ES^0$ is a weak equivalence in $\mathcal{C}$. Hence the weak equivalences of $\mathcal{C}_E$ are precisely the $L_ES^0$–equivalences.*

*Proof.* We first show the statement for a spectral model category $\mathcal{D}$. Recall the model category $L_S\mathcal{D}$ for $S$ the set $I_{\mathcal{D}}\Box J_E$ from the previous section. We will show that the $S$–equivalences are precisely the $L_E S^0$–equivalences of $\mathcal{D}$.

Every map in the set $S$ is an $L_E S^0$–equivalence, hence every $S$–equivalence is a $L_E S^0$–equivalence. Now take some $L_E S^0$–equivalence $f\colon X \to Y$ in $\mathcal{D}$. The map

$$X \to X \wedge^L L_E S^0$$

is a $T$–equivalence with

$$T = \{X \wedge^L f \mid X \in \mathcal{D},\ f \text{ is an } E\text{–equivalence of spectra}\}$$

defined earlier in this section. Hence it is an $S$–equivalence. Thus the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \wedge^L L_E S^0 \arrow[r, "f\wedge^L L_E S^0"] & Y \wedge^L L_E S^0 \end{tikzcd}$$

shows that $f$ is $S$–equivalent to a weak equivalence in $\mathcal{D}$. Weak equivalences in $\mathcal{D}$ are in particular $S$–equivalences, so by the 2–out–of–3 axiom of model categories, $f$ must be an $S$–equivalence.

To move this result from a spectral $\mathcal{D}$ to a general $\mathcal{C}$ we use a similar argument to that of the second point of Theorem 6.1. The Quillen equivalence $(\Sigma^\infty, \Omega^\infty)$ takes the $L_E S^0$–equivalences of $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$ bijectively to the $L_E S^0$–equivalences of $\mathcal{C}$. It follows that the $L_E S^0$–equivalences of $\mathcal{C}$ are precisely the weak equivalences of $\mathcal{C}_E$. □

The following corollary shows that stable $E$–familiarisation restricts to $E$–localisation in the case of spectra. This shows that the notion of $\mathcal{C}_E$ is indeed a good candidate for an analogue of $E$–localisation of a general $\mathcal{C}$.

**Corollary 6.8.** *Consider the category of modules over a ring spectrum $R$. Then*

$$(R\text{–mod})_E = L_E(R\text{–mod})$$

*where the right hand side is the naive localisation of* $R$–mod. *It has weak equivalences those maps of $R$–modules which forget to $E$–equivalences of spectra and the same cofibrations as* $R$–mod. *In particular*

$$\mathcal{S}_E = L_E\mathcal{S}.$$

*Proof.* We start by noting that $L_E(R\text{–mod})$ is equal to the model structure of $R$-modules in $L_E\mathcal{S}$ from [SS00, Theorem 4.1]. Hence this model structure has generating sets of cofibrations and acyclic cofibrations $R \wedge I_{\mathcal{S}}$ and $R \wedge J_E$ [SS00, Lemma 2.3].

We claim that every map in $R \wedge J_E$ is a $T'$–equivalence, where $T'$ is from Theorem 6.1. We know that the domains of $J_E$ are cofibrant, hence $R \wedge j$ is weakly equivalent to $R \wedge^L j$ for any $j \in J_E$. Thus the claim holds. It follows that every acyclic cofibration of $L_E(R\text{–mod})$ is a weak equivalence (and also a cofibration) of $(R\text{–mod})_E$.

We must now show the converse. Every acyclic cofibration of $(R\text{–mod})_E$ is an $E$–equivalence of underlying spectra and hence is an acyclic cofibration of $L_E(R\text{–mod})$. Thus the two model structures agree. □

We can now give a simple proof that stable $E$–familiarisation preserves Quillen equivalences.

**Proposition 6.9.** *Let $\mathcal{C}$ and $\mathcal{E}$ be proper, cellular and stable model categories such that the domains of their generating cofibrations are cofibrant. Let*

$$F : \mathcal{C} \rightleftarrows \mathcal{E} : G$$

*be a Quillen equivalence. Then there is a Quillen equivalence between the $E$–familiarised model categories*

$$F : \mathcal{C}_E \rightleftarrows \mathcal{E}_E : G.$$

*Proof.* Composing $F$ with the identity on $\mathcal{E}$ gives us a left Quillen functor

$$F : \mathcal{C} \to \mathcal{E}_E$$

and $\mathcal{E}_E$ is of course stably $E$–familiar. Hence by the universal property of $\mathcal{C}_E$ proved in Theorem 6.1 we have a left Quillen functor $F : \mathcal{C}_E \to \mathcal{E}_E$. We now need to show that gives us a Quillen equivalence. We do so using Proposition 5.4 and the method of the second part of the proof of Theorem 6.1.

Let $T$ be the class of maps

$$T = \{A \wedge^L f \mid A \in \mathcal{C},\ f \text{ is an } E\text{–equivalence of spectra}\}.$$

Similarly, let $T'$ be the class of maps

$$T' = \{B \wedge^L f \mid B \in \mathcal{E},\ f \text{ is an } E\text{–equivalence of spectra}\}.$$

Then $\mathcal{C}_E = L_T\mathcal{C}$ and $\mathcal{E}_E = L_{T'}\mathcal{E}$. Let $\mathbb{L}F$ and $\mathbb{R}G$ denote the left and right derived functors of $F$ and $G$ respectively. By [Hir03, Theorem 3.3.20], the adjunction $(F, G)$ induces a Quillen equivalence between $L_T\mathcal{C}$ and $L_{\mathbb{L}F(T)}\mathcal{E}$. But the set $\mathbb{L}F(T)$ is isomorphic in $\mathrm{Ho}\,\mathcal{E}$ to the set $T'$ because Quillen equivalences induce equivalences of $\mathrm{Ho}(\mathcal{S})$–module categories [Len12, Theorem 6.3]. □

**Remark 6.10.** *One could try to prove an analogue of Proposition 5.3 and show that if $\mathcal{C}$ is monoidal then so is $\mathcal{C}_E$. This would require the adjunction $(\Sigma^\infty, \Omega^\infty)$ at the beginning of this section to be monoidal. However we do not know if this is the case.*

## 7. Examples

Let $\mathcal{C}$ be a spectral model category, such that the domains of its generating cofibrations are cofibrant. (Recall from [BR14, Theorem 7.2] that any stable, proper and cellular model category is Quillen equivalent to a spectral one.) Assume that $\mathcal{C}$ has a set of compact generators for its homotopy category, [SS03, Definition 2.1.2]. Schwede and Shipley prove in the above–mentioned paper that any such $\mathcal{C}$ is Quillen equivalent to a category mod–$\mathcal{E}$ where $\mathcal{E}$ can be thought of as a "ring spectrum with several objects". In the case of $\mathcal{C}$ having a single compact generator, $\mathcal{E}$ is simply a ring spectrum.

Let us briefly recap some of the definitions and constructions of that result. Let $\mathcal{G}$ denote the set of generators of $\mathcal{C}$. Then the $\mathcal{S}$–enriched category $\mathcal{E}$ is simply defined as the full $\mathcal{S}$–enriched subcategory of $\mathcal{C}$ with objects $\mathcal{G}$. An object $M \in$ mod–$\mathcal{E}$ consists of a spectrum $M(G)$ for each $G \in \mathcal{G}$ plus morphisms of spectra

$$\mathcal{E}(G', G) \wedge M(G) \longrightarrow M(G') \ \text{ for } \ G, G' \in \mathcal{G}$$

satisfying certain coherence conditions. By adjunction, such an $M$ is the same as a contravariant spectral functor from $\mathcal{E}$ to $\mathcal{S}$. A standard example of an object of mod–$\mathcal{E}$ is given by the spectral functor $\mathcal{E}(-, G)$ for fixed $G \in \mathcal{G}$. The model structure on mod–$\mathcal{E}$ is has weak equivalences and fibrations defined objectwise [SS03, Theorem A.1.1]. This means that a natural transformation $f : M \longrightarrow N$ is a weak equivalence or a fibration if and only if

$$f_G : M(G) \longrightarrow N(G)$$

is so for each $G \in \mathcal{G}$. Theorem 3.9.3 of [SS03] then describes a Quillen equivalence

$$\mathrm{Hom}(\mathcal{G}, -) : \mathcal{C} \leftrightarrows \text{mod–}\mathcal{E} : - \wedge_{\mathcal{E}} \mathcal{G}$$

for spectral $\mathcal{C}$.

This is a highly useful description of a stable model category and we would like to obtain a description of the $E$–familiarisation $\mathcal{C}_E$ of $\mathcal{C}$ in terms of mod–$\mathcal{E}$. We note that this is a rather special case as not every stable model category has a set of compact generators [HS99, Corollary B.13].

By Proposition 6.9 we know that $\mathcal{C}_E$ and (mod–$\mathcal{E})_E$ are Quillen equivalent, so we shall find another description of (mod–$\mathcal{E})_E$.

Since mod–$\mathcal{E}$ is a spectral model category, it is easily seen that (mod–$\mathcal{E})_E$ is given by $L_S$ mod–$\mathcal{E}$ as in Proposition 5.2. Recall that $S = I \square J_E$ for $I$ the set of generating cofibrations for mod–$\mathcal{E}$. Hence in (mod–$\mathcal{E})_E$ any map of the form below is a weak equivalence.

$$\mathcal{E}(-,G) \wedge (i \square j)$$

In the above, $G$ is a cofibrant and fibrant replacement of one of the compact generators for $\mathcal{C}$, $i$ is a generating cofibration for $\mathcal{S}$ and $j$ is a generating acyclic cofibration for $L_E\mathcal{S}$.

We can make another model structure on mod–$\mathcal{E}$ by taking the same cofibrations as before, but taking the generating set of acyclic cofibrations to be those maps of the form

$$\mathcal{E}(-,G) \wedge j$$

for $G$ a generator and $j$ a generating acyclic cofibration for $L_E\mathcal{S}$. We shall call this set of maps $K$ and let mod–$\mathcal{E}_K$ denote the corresponding model structure. One can either check directly that these sets give a model structure or one can alter [SS03, Theorem A.1.1] to use $L_E\mathcal{S}$ instead of $\mathcal{S}$.

We claim that this model structure equals the model structure of (mod–$\mathcal{E})_E$. An element of $K$ can be described as

$$\mathcal{E}(-,G) \wedge \big((* \to S^0) \square j\big) .$$

Hence every element of $K$ is an acyclic cofibration of (mod–$\mathcal{E})_E$. Conversely, mod–$\mathcal{E}$ equipped with this new model structure is stably $E$–familiar. Hence the identity functor (mod–$\mathcal{E})_E \to$ mod–$\mathcal{E}_K$ is a left Quillen functor. Hence every acyclic cofibration of (mod–$\mathcal{E})_E$ is an acyclic cofibration of mod–$\mathcal{E}_K$. Thus these two model structures have the same cofibrations and acyclic cofibrations. We have therefore shown the following.

**Proposition 7.1.** *The model category* (mod–$\mathcal{E})_E$ *is the category of contravariant spectral functors from $\mathcal{E}$ to $L_E\mathcal{S}$, equipped with the model structure where fibrations and weak equivalences are defined objectwise. Thus the fibrant objects are those functors $M$ such that $M(G)$ is fibrant in $\mathcal{S}$ and $E$–local for all $G \in \mathcal{E}$.* □

Consider the case where $\mathcal{C}$ has a single compact generator. Following the above we can replace this by a category of functors to $\mathcal{S}$. Indeed, [SS03, Theorem 3.1.1] states that $\mathcal{C}$ is Quillen equivalent to the category of $R$–modules, mod–$R$, for some ring spectrum $R$. In this case, the above proposition recovers the result of Corollary 6.8.

## 8. $E$–familiarisation and homotopy pushouts

We want to give another description of $\mathcal{C}_E$ via a universal property. We will relate $\mathcal{C}_E$ to a pushout of model categories. While the pullback of model categories is well–understood, [Ber11], the pushout is more complicated and is not often used. Roughly speaking, the homotopy pushout of a corner diagram of Quillen adjunctions

$$\mathcal{C} \leftrightarrows \mathcal{D} \rightleftarrows \mathcal{E}$$

is supposed to be a model category $\mathcal{P}$ that satisfies a universal property analogous to the pushout of a diagram within a category. Unfortunately, the homotopy–theoretic pushout construction is rather delicate and its existence and description not always clear.

However there is a special case where we can construct pushouts of model categories and verify that they have the correct universal property. By working in a particular context, we avoid the general question of whether homotopy pushouts of model categories exist in general.

Let $\mathcal{M}_2$ be a left Bousfield localisation of $\mathcal{M}_1$ at a class of maps $W$. Without loss of generality we assume that the maps in $W$ are morphisms between cofibrant objects. (If the elements of $W$ did not satisfy this, one can replace them with weakly equivalent morphisms between cofibrant objects. This would then give rise to the same Bousfield localisations.) In particular, this gives us a Quillen pair

$$\mathrm{Id}\colon \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 = L_W\mathcal{M}_1 : \mathrm{Id}$$

Assume that we have a Quillen adjunction

$$F : \mathcal{M}_1 \rightleftarrows \mathcal{N}_1 : G.$$

We are now going to discuss the homotopy pushout of the corner diagram below for this special case

$$L_W\mathcal{M}_1 = \mathcal{M}_2 \leftrightarrows \mathcal{M}_1 \rightleftarrows \mathcal{N}_1.$$

**Definition 8.1.** *The homotopy pushout of the above diagram is defined, if it exists, as the Bousfield localisation $L_{\mathbb{L}FW}\mathcal{N}_1$ of $\mathcal{N}_1$. Here, $\mathbb{L}F$ denotes the left derived functor of $F$.*

To justify this definition we need to see that $\mathcal{N}_2 = L_{\mathbb{L}FW}\mathcal{N}_1$ (provided it exists) satisfies the desired properties that a homotopy pushout is supposed to have. First we note that by [Hir03, Theorem 3.3.20] $F$ and $G$ induce a Quillen adjunction

$$F : \mathcal{M}_2 \rightleftarrows \mathcal{N}_2 : G$$

Assume that there is a model category $\mathcal{D}$ with Quillen adjunctions

$$\begin{array}{rcl} \mathcal{M}_2 & \rightleftarrows & \mathcal{D} \\ F' : \mathcal{N}_1 & \rightleftarrows & \mathcal{D} : G' \end{array}$$

such that in the diagram below, the two different composites of left adjoints from $\mathcal{M}_1$ to $\mathcal{D}$ agree up to natural isomorphism.

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, shift left] \arrow[d, shift left] & \mathcal{N}_1 \arrow[l, shift left] \arrow[dd, bend left, shift left] \\
\mathcal{M}_2 \arrow[u, shift left] \arrow[dr, bend right, shift left] & \\
& \mathcal{D} \arrow[ul, bend left, shift left] \arrow[uu, bend right, shift left]
\end{tikzcd}$$

Because the vertical functors in the square below are simply identity functors it follows immediately that we may add $\mathcal{N}_2$ and obtain a commutative diagram of adjoint pairs.

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, shift left] \arrow[d, shift left] & \mathcal{N}_1 \arrow[l, shift left] \arrow[d, shift left] \arrow[dr, bend left, shift left] & \\
\mathcal{M}_2 \arrow[u, shift left] \arrow[r, shift left] \arrow[drr, bend right, shift left] & \mathcal{N}_2 \arrow[l, shift left] \arrow[u, shift left] \arrow[dr, dotted, shift left] & \\
& & \mathcal{D} \arrow[ull, bend left, shift left] \arrow[ul, dotted, shift left] \arrow[uul, bend right, shift left]
\end{tikzcd}$$

We must check that the adjunction below is a Quillen adjunction.

$$F' : \mathcal{N}_2 \rightleftarrows \mathcal{D} : G'$$

The model category $\mathcal{N}_2$ is the Bousfield localisation of $\mathcal{N}_1$ with respect to the class of maps $Ff$ where $f$ is a weak equivalence between cofibrant objects of $\mathcal{M}_2$. Thus $(F' \circ F)(f)$ is a weak equivalence in $\mathcal{D}$. This means that $F'$ uniquely factors over $\mathcal{N}_2$. Furthermore, by construction, $\mathcal{N}_2$, if it exists, is unique up to Quillen equivalence.

Recall that the stable $E$–familiarisation $\mathcal{C}_E$ satisfies the following universal property. Given a left Quillen functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ with $\mathcal{D}$ stably $E$–familiar, $F$ also gives rise to a left Quillen functor $C_E \longrightarrow \mathcal{D}$ via

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "id"'] & \mathcal{D}. \\ \mathcal{C}_E \arrow[ur] & \end{tikzcd}$$

This fact allows us to relate $\mathcal{C}_E$ and certain homotopy pushouts. Let $X \in \mathcal{C}$ be fibrant and cofibrant. Then we have a Quillen adjunction

$$X \wedge - : \mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Map}_{\mathcal{C}}(X, -).$$

Using Definition 8.1 we can read off the following for a proper and cellular stable model category $\mathcal{C}$.

**Lemma 8.2.** *The homotopy pushout $\mathcal{P}_X$ of the diagram*

$$L_E\mathcal{S} \leftrightarrows \mathcal{S} \rightleftarrows \mathcal{C}$$

*exists and is the Bousfield localisation of $\mathcal{C}$ with respect to the set of maps below, where $J_E$ is the set of generating acyclic cofibrations of $L_E\mathcal{S}$.*

$$X \wedge^L J_E = \{X \wedge^L j \mid j \in J_E\}$$

□

So in particular we know that this homotopy pushout exists. Because $\mathcal{C}_E$ is stably $E$–familiar we have a commutative square of Quillen adjunctions

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift left] \arrow[d, shift left] & \mathcal{C} \arrow[l, shift left] \arrow[d, shift left] \\ L_E\mathcal{S} \arrow[r, shift left] \arrow[u, shift left] & \mathcal{C}_E \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd}$$

By the universal property of $\mathcal{P}_X$, there is a Quillen adjunction $\mathcal{P}_X \rightleftarrows \mathcal{C}_E$ for each $X$. We can show that $\mathcal{C}_E$ is the "closest" model category to those pushouts in the following sense.

**Theorem 8.3.** *Let $\mathcal{C}$ be a stable, proper and cellular model category, such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. The Quillen adjunction*

$$\mathcal{C} \rightleftarrows \mathcal{C}_E$$

*factors over*

$$\mathcal{P}_X \rightleftarrows \mathcal{C}_E$$

*for all fibrant–cofibrant $X \in \mathcal{C}$. If there is any other stable $\mathcal{D}$ with a Quillen adjunction*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

*that factors over*

$$\mathcal{P}_X \rightleftarrows \mathcal{D}$$

*for all fibrant–cofibrant $X$, then $(F, G)$ also factors over $\mathcal{C}_E$.*

*Proof.* The pushout $\mathcal{P}_X$ is defined as the Bousfield localisation of $\mathcal{C}$ at the set of maps $X \wedge^L j$ with $j \in J_E$. By Proposition 5.4 we know that $\mathcal{C}_E$ is the localisation of $\mathcal{C}$ at the class of maps of the form $X \wedge^L f$ for $f$ an $E$–equivalence of spectra. Thus we see that for every $X \in \mathcal{C}$ the identity gives us a Quillen adjunction

$$\mathrm{Id} : \mathcal{P}_X \rightleftarrows \mathcal{C}_E : \mathrm{Id}$$

because every weak equivalence in $\mathcal{P}_X$ is also a weak equivalence in $\mathcal{C}_E$.

If the given Quillen adjunction $(F, G)$ induces a Quillen adjunction

$$F : \mathcal{P}_X \rightleftarrows \mathcal{D} : G,$$

then $F$ sends all morphisms of the form $X \wedge^L j$, for $j \in J_E$, to weak equivalences in $\mathcal{D}$. Hence $F$ also sends all maps of the form $X \wedge^L f$, for $f$ an $E$–equivalence of spectra, to weak equivalences in $\mathcal{D}$.

If $(F, G)$ gives such Quillen adjunctions for all fibrant–cofibrant $X$ then it must send any map of the form $X \wedge^L f$ with $X$ fibrant–cofibrant and $f$ an $E$–equivalence of spectra to a weak equivalence in $\mathcal{D}$. Thus it induces a Quillen adjunction

$$\mathcal{C}_E \rightleftarrows \mathcal{D}$$

by Theorem 6.1, which is what we wanted to prove. □

## 9. Modular rigidity for $E$–local spectra

We can show that stable frames encode all homotopical information of the $E$–local stable homotopy category. The triangulated structure of $\mathrm{Ho}(L_E\mathcal{S})$ alone is not sufficient for this: given just a triangulated equivalence

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

for a stable model category $\mathcal{C}$ does not imply in general that $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent. In fact, Quillen equivalence can only be deduced from a triangulated equivalence of homotopy categories in some very special cases. To this date, the only nontrivial cases known of this 'rigidity' are the stable homotopy category itself [Sch07] and the case $E = K_{(2)}$ [Roi07]. However, if we do not only have a triangulated equivalence as above but also assume that this equivalence is a $\mathrm{Ho}(\mathcal{S})$–module equivalence, we can show that $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent.

We now give a more general version of [BR11, Theorem 9.5], in particular the assumption that $E$ is smashing is no longer required.

**Theorem 9.1.** *Let $\mathcal{C}$ be a stable model category. Assume we have an equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

*then $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent if and only if $\Phi$ is an equivalence of $\mathrm{Ho}(\mathcal{S})$–module categories.*

*Proof.* The "only if" part is true by [Len12, Theorem 6.3]: a Quillen equivalence induces a $\mathrm{Ho}(\mathcal{S})$–module equivalence.

Now let us assume that we have a $\mathrm{Ho}(\mathcal{S})$–module equivalence

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

It follows that $\Phi^{-1}$ induces a weak equivalence of homotopy mapping spectra

$$\Phi^{-1} : \mathbb{R}\,\mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \mathbb{R}\,\mathrm{Map}_{L_E\mathcal{S}}(\Phi^{-1}X, \Phi^{-1}Y)$$

for $X, Y \in \mathcal{C}$. The right–hand–side is an $E$–local spectrum as $L_E\mathcal{S}$ is stably $E$–familiar. Hence every homotopy mapping spectrum of $\mathcal{C}$ is $E$–local, so $\mathcal{C}$ is stably $E$–familiar by [BR11, Theorem 7.8].

Thus for fibrant and cofibrant $X \in \mathcal{C}$ , the Quillen functor

$$X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$$

factors over $L_E\mathcal{S}$ as a Quillen functor

$$X \wedge - : L_E\mathcal{S} \longrightarrow \mathcal{C}.$$

Now let $X$ be a cofibrant–fibrant replacement of $\Phi(S^0)$. Because $\Phi$ is a $\mathrm{Ho}(\mathcal{S})$–module equivalence we see that

$$X \wedge^L (-) = \Phi(S^0) \wedge^L (-) = \Phi(S^0 \wedge^L -) = \Phi(-).$$

This means that $\Phi$ is derived from a Quillen functor. This Quillen functor must therefore be a Quillen equivalence, which is what we wanted to prove. □

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